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Relativistic Orbital Perturbations in a Weak Gravitational Field

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With the general third-order equations of motion for a test particle, Synge's third-order orbital equations at great distance in the weak gravitational field generated by a massive body are derived. The body has an axis of symmetry around which is rotating steadily. The results found for the advance of perihelion using first integrals of motion for the general equations show that the effect due to the inner stress of the body can be derived for orbits with inclination with respect to the equator of the body. Then, by means of the variation of the parameters method, we obtain with the equations at great distance the corresponding perturbations on the elements of such orbits in the field considered. These perturbations result to be of second order with regard to the mass of the body (the basis of the approximation).

1. INTRODUCTION

In a previous paper (Gambi, 1983) Synge's approximation method has been applied to obtain the weak gravitational field of a massive body with an axis of symmetry around which is rotating steadily (Synge, 1970). The method was carried out to include the second approximation in the field, which is enough to third-order equations of motion. This means that terms in it of order m^2 are retained as significant and that there is an error of order m^3 in the field equations, m being the mass of the body.

The main difference between Synge's approach and that of Chandrasekhar (1965) and Chandrasekhar and Nutku (1969) lies in the gauge conditions used. Whereas the conditions of these authors lead to Poisson equations and instantaneous potential solutions, Synge's conditions lead to inhomogeneous wave equations and retarded potential solutions as in the scheme of Anderson and Decanio (1975). In this respect, Synge's method is closer to that of Anderson and Decanio although their gauge conditions and corresponding energy pseudotensors are also different.

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In particular, the field obtained in Gambi (1983) is more sophisticated than the one obtained in the standard post-Newtonian approximation (Chandrasekhar, 1965) because the $O(m^2)$ terms in the second-order deviations $\gamma_{\mu\nu}$ [(23) below] are explicitly determined and the same happens with the $O(m^{5/2})$ terms of $\gamma_{\mu 4}$.

Starting with Synge's third-order equations of motion, in the present paper the third-order general orbital equations corresponding to this field and also we obtain the corresponding equations for the motion at great distance of the test particle. These equations show that the effect from the inner stress of the massive body on the orbital motion is of order m^2, or to be more precise, of order m^2/r^3, r being the distance from the test particle to the massive body.

When the massive body has an equatorial plane of symmetry, we can find in the case of an equatorial orbit (Gambi, 1985) the diverse contributions to the advance of perihelion of such an orbit only with the help of the two first integrals which correspond to the energy and angular momentum of the test particle. But in the case of an arbitrary orbit (that is to say, with inclination with respect to the equator) it is necessary to return to the general equations. Then, in order to derive the contributions to the variation of the elements of such an orbit due to the inner stress of the body, we use the corresponding orbital equations at great distance and the perturbation theory. The expressions found show that the corrections only appear in the mean anomaly, in the argument of perihelion, and in the longitude of the ascending node.

2. NOTATION AND GENERAL METHOD

For details of Synge's method the reader is referred to Synge (1970). Here we shall simply give a brief outline of the method. For a metric

$$g_{ab} = \delta_{ab} + \gamma_{ab} \tag{1}$$

we have the linear part of the Einstein tensor L_{ab} defined by

$$\begin{aligned} L_{ab} = {} & \tfrac{1}{2}(\gamma_{ab,cc} + \gamma_{cc,ab} - \gamma_{ac,cb} - \gamma_{bc,ca}) \\ & -\tfrac{1}{2}\delta_{ab}(\gamma_{cc,dd} - \gamma_{cd,cd}) \end{aligned} \tag{2}$$

and the truncated Einstein tensor $\hat{G}_{ab}$ by

$$\hat{G}^{ab} = G^{ab} - L_{ab} \tag{3}$$

the comma denoting partial differentiation. Latin indices take the values 1, 2, 3, 4 and greek indices 1, 2, 3.

We take the point of view that the energy tensor T^{ab} is given and from it we generate a sequence of metrics

$$\underset{M}{g}_{ab} = \delta_{ab} + \underset{M}{\gamma}_{ab} \qquad (M = 0, 1, 2, \ldots, N) \tag{4}$$

by the recurrence formula

$$\underset{M}{\gamma}^{*}_{ab} = 2\kappa K^{ab}_{rs} \underset{M-1}{H}^{rs} \qquad (M = 1, 2, \ldots, N),\ \underset{0}{\gamma}_{ab} = 0\ (\kappa = 8\pi) \tag{5}$$

where

$$\underset{M}{\gamma}^{*}_{ab} = \underset{M}{\gamma}_{ab} - \tfrac{1}{2}\delta_{ab} \underset{M}{\gamma}_{dd} \qquad (M = 0, 1, 2, \ldots, N) \tag{6}$$

$$\underset{M}{H}^{ab} = T^{ab} + \kappa^{-1} \underset{M}{\hat{G}}^{ab} \qquad (M = 0, 1, 2, \ldots, N) \tag{7}$$

with $\underset{M}{\hat{G}}^{ab} = \hat{G}^{ab}(\underset{M}{\gamma})(M = 0, 1, \ldots, N)$ and

$$K^{ab}_{rs} = -\delta_{ar}\delta_{bs}J + J(\delta_{ar}D_{bs} + \delta_{bs}D_{ar} - \delta_{ab}D_{rs})J \tag{8}$$

D_a and D_{ab} indicate partial derivatives of first and second orders, respectively, and J is defined by

$$Jf(\mathbf{x}, t) = -(4\pi)^{-1} \int f(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)|\mathbf{x} - \mathbf{x}'|^{-1}\, d_3x' \tag{9}$$

so that

$$\Box Jf = J\Box f = f \qquad (\Box = D_{aa}) \tag{10}$$

From (5) and (8) each term of the sequence (4) satisfies the gauge conditions

$$\underset{M}{\gamma}^{*}_{ab,b} = 0 \qquad (M = 0, 1, 2, \ldots, N) \tag{11}$$

This simplifies $\underset{M}{L}_{ab}$ given by (2) to read $\underset{M}{L}_{ab} = \frac{1}{2}\Box\ \underset{M}{\gamma}^{*}_{ab}$ and then we have

$$\underset{M}{L}_{ab} = \kappa\left[-\underset{M-1}{H}^{ab} + J\left(D_{bc}\underset{M-1}{H}^{bc} + D_{ac}\underset{M-1}{H}^{bc} - \delta_{ab}D_{cd}\underset{M-1}{H}^{cd}\right)\right]$$
$$(M = 1, 2, 3, \ldots, N) \tag{12}$$

In order to introduce approximations we introduce a small parameter k in terms of which the energy tensor components can be expressed, and provisionally we assume

$$T^{ab} = O(k) \tag{13}$$

Then we have from (5)

$$\underset{M}{\gamma}_{ab} = \underset{M-1}{\gamma}_{ab} + O(k^{\mu}) \qquad (M = 1, 2, 3, \ldots, N) \tag{14}$$

and

$$\underset{M}{\hat{G}}{}^{ab} = \underset{M-1}{\hat{G}}{}^{ab} + O(k^{M+1}), \qquad (M = 1, 2, 3, \ldots, N) \tag{15}$$

If we stop at the Nth term, $\underset{N}{\gamma}_{ab}$, and impose on T^{ab} the equations

$$\underset{N-1}{H}{}^{ab}, b = 0 \tag{16}$$

then $\underset{N}{g}_{ab}$ satisfy

$$\underset{N}{G}{}^{ab} = -\kappa T^{ab} + O(k^{N+1}) \tag{17}$$

so that the field equations are satisfied with an error $O(k^{N+1})$ and we have

$$\underset{N}{\gamma}{}^{*}_{ab} = -2\kappa J \underset{N-1}{H}{}^{ab} \tag{18}$$

The equations (16) are equivalent to

$$T^{ab} \underset{N}{|}_{b} = O(k^{N+1}) \tag{19}$$

where the N below the stroke indicates covariant derivative with respect to $\underset{N-1}{g}_{ab}$. By this they are called by Synge equations of motion in Nth approximation.

With this scheme we can consider, in a systematic way, approximate solutions of Einstein's field equations and equations of motion to any degree of accuracy we wish. Note that for equations of motion in Nth approximation we only need the metric components to $O(k^{N-1})$. Later we shall relax this requirement when we consider the separate components of these equations and the differing orders of magnitude for the components of the energy tensor T^{ab}. In the present work we shall consider equations of motion in third approximation.

3. DESCRIPTION OF THE MODEL

The weak gravitational field which we consider is generated by a massive body rotating steadily around its axis of symmetry.

We assume as topology of space-time the one of a Euclidean 4-space. We use rectangular Cartesian coordinates x_μ and imaginary time $x_4 = it$. Choosing as axis of symmetry the axis Ox_3, we have

$$u_1 = -\frac{u}{r}x_2, \qquad u_2 = \frac{u}{r}x_1, \qquad u_3 = 0, \qquad u_4 = i \tag{20}$$

$$\frac{\partial u_\alpha}{\partial t} = 0, \qquad \frac{\partial \rho}{\partial t} = 0 \tag{20'}$$

$$u_\alpha = O(k^{1/2}), \qquad \rho = O(k) \tag{20''}$$

where u_α and u are, respectively, the Eulerian 3-velocity and the velocity of the body satisfying $u = u(r, x_3)(r^2 = x_1^2 + x_2^2)$, and $\rho = \rho(r, x_3)$ is its Eulerian density. k is a small dimensionless parameter of the same order of the mass of the body measured in seconds which constitutes the basis of the approximation. All magnitudes are measured in seconds. For details see Gambi (1983).

Let I be the history of the world tube corresponding to the body and E the part of space-time exterior to I. We take as Eulerian description of the model the following

$$\begin{gathered} T^{\mu\nu} = \rho u_\mu u_\nu - S_{\mu\nu}, \qquad T^{\mu 4} = i\rho u_\mu, \qquad T^{44} = -\rho < 0 \qquad \text{in } I \\ T^{ab} = 0 \qquad \text{in } E \end{gathered} \tag{21}$$

and we call to $S_{\mu\nu}$ the Eulerian stress of the body. $S_{\mu\nu}$ is $O(k^2)$. If T^{ab} satisfy the equations of motion

$$\rho\frac{du_\mu}{dt} - S_{\mu\nu,\nu} = \rho V_{,\mu} + O(k^3) \tag{22}$$

$$\frac{d\rho}{dt} + \rho u_{\mu,\mu} = O(k^{7/2}) \tag{22'}$$

then the second-order metric deviations γ_{ab} with respect to the Minkowskian metric δ_{ab} are given by

$$\begin{aligned} \gamma_{\mu\nu} &= 2(V + V^2)\delta_{\mu\nu} - 2\kappa\tilde{J}[(\rho u_\mu u_\nu)^*] \\ &\quad + 4\tilde{J}[\tfrac{1}{2}\kappa S^*_{\mu\nu} + (V^2)_{,\mu\nu} - V_{,\mu}V_{,\nu}] + O(k^3) \\ \gamma_{\mu 4} &= -2\kappa i\tilde{J}(\rho u_\mu) - 8i\tilde{J}(V_{,\sigma}W_{\sigma,\mu} - V_{1\mu\sigma}W_\sigma \\ &\quad + W_\mu \Delta V - V\Delta W_\mu) + O(k^{7/2}) \\ \gamma_{44} &= -2(V - V^2) - \kappa\tilde{J}S_{\sigma\sigma} + \kappa\tilde{J}(\rho u_\sigma u_\sigma) + O(k^3) \end{aligned} \tag{23}$$

where

$$\tilde{J}[f(\mathbf{x}, t)] = -(4\pi)^{-1}\int f(\mathbf{x}', t)|\mathbf{x}-\mathbf{x}'|^{-1}\, d_3x' \tag{24}$$

$$S^*_{\mu\nu} = S_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}S_{\sigma\sigma}, \qquad (\rho u_\mu u_\nu)^* = (\rho u_\mu u_\nu) - \tfrac{1}{2}\delta_{\mu\nu}(\rho u)^2 \tag{24'}$$

$$V = -4\pi\tilde{J}(\rho) \tag{24''}$$

$$W_\mu = -4\pi\tilde{J}(\rho u_\mu) \tag{24'''}$$

We then have a universe which contains a body of arbitrary shape with the only restriction that it possesses an axis of symmetry around which is rotating with small velocity, $u_\mu = O(k^{1/2})$, in his own gravitational field. The density ρ is small, $O(k)$. The energy tensor is given by (21), where $S_{\mu\nu}$ is the Newtonian stress under gravity. The metric tensor is

$$g_{ab} = \delta_{ab} + \gamma_{ab} \tag{25}$$

where γ_{ab} is given by (23).

It may be shown that the corresponding field at great distance is given by

$$\begin{aligned} &g_{\mu\nu} = \delta_{\mu\nu} + 2\delta_{\mu\nu}(m+m')r^{-1}, \qquad g_{44} = 1 - 2(m+m')r^{-1} \\ &g_{14} = 2ix_2 r^{-3}L_3, \qquad g_{24} = -2ix_1 r^{-3}L_3, \qquad g_{34} = 0 \end{aligned} \tag{26}$$

where m is the mass of the body, L_3 its angular momentum with respect to Ox_3, and $m' = \int \rho V\, d_3x'$ is the mass of the field.

4. THE ORBITAL EQUATIONS

In order to derive the orbital equations in the field considered we begin considering Synge's equations of motion in third approximation [cf. equations (1.61), (1.62), and (1.63) of Synge, 1970]:

$$\begin{aligned} T^{\mu b}_{,b} &= \rho\dot{u}_\mu + u_\mu(\dot{\rho} + \rho u_{\sigma,\sigma}) - S_{\mu\nu,\nu} = \rho V_{,\mu} + Y_\mu + O(k^4) \\ -iT^{4b}_{,b} &= \dot{\rho} + \rho u_{\sigma,\sigma} = -\rho V_{,t} + Z + O(k^{9/2}) \end{aligned} \tag{27}$$

where

$$\begin{aligned} Y_\mu = {} & T^{\sigma\sigma}V_{,\mu} - 4T^{\mu\nu}V_{,\nu} + 4\rho u_\nu(W_{\mu,\nu} - W_{\nu,\mu} - \delta_{\mu\nu}V_{,t}) \\ & + \rho D_\mu(\tfrac{1}{2}D_t^2 I_2\rho - 2V^2 + k_{\sigma\sigma}) + 4\rho D_t W_\mu = O(k^3) \end{aligned} \tag{28}$$

$$\begin{aligned} Z = {} & -T^{\sigma\sigma}V_{,t} - 4T^{\mu\nu}W_{\mu,\nu} + \rho u_\nu D_\nu(4K_{\sigma\sigma} - V^2 - 2N) \\ & + \rho D_t[3K_{\sigma\sigma} - 2N - V^2 - \tfrac{1}{2}D_t^2 I_2\rho - I_1(\rho V_{,t})] \\ & - 2\rho V_{,\sigma}W_\sigma + \tfrac{1}{3}\rho D_t^2 I_1(T^{\sigma\sigma} + \tfrac{1}{2}\rho V) = O(k^{7/2}) \end{aligned} \tag{28'}$$

and $K_{\sigma\sigma}=I_0T^{\sigma\sigma}$, $N=I_0(\rho V)$, where

$$I_nf(\mathbf{x}, t)=\int f(\mathbf{x}', t)|\mathbf{x}-\mathbf{x}'|^{n-1}\, d_3x' \qquad (n=0,1,2) \tag{28''}$$

Now we take the case in which the motion of only two bodies is considered and we suppose that one of the bodies is very small with respect to the other. Then we can neglect in (27) the self-potentials in the small body. Furthermore as the field (25) is stationary, all terms in (27) in which derivatives with respect to t appear vanish. Then, with these assumptions, equations (27) are reduced to

$$\begin{aligned}\dot{u}_\rho &= V_{,\rho}+\rho^{-1}T^{\sigma\sigma}V_{,\rho}-4\rho^{-1}T^{\rho\nu}V_{,\nu}\\ &\quad +4u_\nu(W_{\rho,\nu}-W_{\nu,\rho})-4VV_{,\rho}+K_{\sigma\sigma,\rho}+O(k^3)\end{aligned} \tag{29}$$

and from here we have

$$\begin{aligned}\dot{u}_\rho &= V_{,\rho}+V_{,\rho}(u^2-4V)-4u_\rho\dot{V}+P_{,\rho}-Q_{,\rho}\\ &\quad +4(W_{\rho,\mu}-W_{\mu,\rho})u_\mu+O(k^3)\end{aligned} \tag{30}$$

where

$$P=\tfrac{1}{2}\kappa\tilde{J}S_{\sigma\sigma} \tag{31}$$

and

$$a=\tfrac{1}{2}\kappa\tilde{J}(\rho u_\sigma u_\sigma) \tag{31'}$$

To be sure that (30) are the orbital equations we must verify the geodesic hypothesis. In order to do this let us make no initial assumption about the character of the field. Accordingly with the geodesic hypothesis, orbits satisfy the equations

$$\ddot{x}_\rho+\Gamma^\rho_{mn}\dot{x}_m\dot{x}_n=\lambda\dot{x}_\rho, \qquad \Gamma^4_{mn}\dot{x}_m\dot{x}_n=i\lambda \tag{32}$$

where Γ^h_{mn} are the Christoffel symbols of second kind, λ is a Lagrange multiplier, and

$$\dot{x}_\rho=u_\rho=O(k^{1/2}) \tag{33}$$

The first three equations (32) are equivalent to

$$\dot{u}_\rho+\Gamma^\rho_{\mu\nu}u_\mu u_\nu+2i\Gamma^\rho_{\mu4}u_\mu-\Gamma^\rho_{44}=\lambda u_\rho \tag{34}$$

If the field is stationary it is easy to see that

$$\begin{aligned}\Gamma^\rho_{\mu\nu}&=\tfrac{1}{2}g^{\rho\alpha}[g_{\mu\alpha,\nu}+g_{\nu\alpha,\mu}-g_{\mu\nu,\alpha}]+\tfrac{1}{2}g^{\rho4}[g_{\mu4,\nu}+g_{\nu4,\mu}]\\ \Gamma^\rho_{\mu4}&=\tfrac{1}{2}g^{\rho\alpha}[g_{4\alpha,\mu}-g_{\mu4,\alpha}]+\tfrac{1}{2}g^{\rho4}g_{44,\mu}\\ \Gamma^\rho_{44}&=-\tfrac{1}{2}g^{\rho\alpha}g_{44,\alpha}\end{aligned} \tag{35}$$

Then substituting (35) in (34) and supposing that $g_{ab}=\delta_{ab}+\gamma_{ab}$ where $\gamma_{ab}=O(k^2)$ we have by a straightforward calculation

$$\dot{u}_\rho+\tfrac{1}{2}\gamma_{44,\rho}-\tfrac{1}{2}\gamma_{\rho\alpha}\gamma_{44,\alpha}+\tfrac{1}{2}(\gamma_{\mu\rho,\nu}+\gamma_{\nu\rho,\mu}-\gamma_{\mu\nu,\rho})u_\mu u_\nu+i(\gamma_{4\rho,\mu}-\gamma_{4\mu,\rho})u_\mu=\lambda u_\rho \tag{36}$$

Doing the same as before with the fourth equation (32) we get

$$\gamma_{\mu 4,\nu}u_\mu u_\nu+i\gamma_{44,\mu}(u_\mu-\gamma_{44}u_\mu)-\tfrac{1}{2}\gamma_{\rho 4}\gamma_{44,\rho}=i\lambda \tag{37}$$

and from this and (36) we obtain

$$\dot{u}_\rho+\tfrac{1}{2}\gamma_{44,\rho}-\tfrac{1}{2}\gamma_{\rho\alpha}\gamma_{44,\alpha}+\tfrac{1}{2}(\gamma_{\mu\rho,\nu}+\gamma_{\nu\rho,\mu}-\gamma_{\mu\nu,\rho})u_\mu u_\nu+i(\gamma_{4\rho,\mu}-\gamma_{4\mu,\rho})u_\mu=\gamma_{44,\mu}u_\mu u_\rho+O(k^3) \tag{38}$$

Finally carrying (23) to (38) we obtain

$$\begin{aligned}\dot{u}_\rho-V_{,\rho}+2VV_{,\rho}-\tfrac{1}{2}\kappa(\tilde{J}S_{\sigma\sigma})_{,\rho}+\tfrac{1}{2}[\tilde{J}(\rho u^2)]_{,\rho}\\+\tfrac{1}{2}[(2V\delta_{\mu\rho})_{,\nu}+(2V\delta_{\nu\rho})_{;\mu}-(2V\delta_{\mu\nu})_{,\rho}]u_\mu u_\nu\\+i\{[-2\kappa i\tilde{J}(\rho u_\rho)]_{,\mu}+[2\kappa i\tilde{J}(\rho u_\mu)]_{,\rho}\}u_\mu\\=-2V_{,\mu}u_\mu u_\rho+O(k^3)\end{aligned} \tag{39}$$

which, on taking into account (31) and (31′), result in the same equations written in (30).

Having demonstrated the geodesic hypothesis, let us now obtain the orbital equations at great distance. We write as usual

$$|\mathbf{x}-\mathbf{x}'|=\frac{1}{r}+\frac{r'}{r^2}\cos\psi+\frac{(r')^2}{r^3}\frac{1}{2}(3\cos^2\psi-1)+O\left(\frac{1}{r^4}\right) \tag{40}$$

where $r=|\mathbf{x}|$, $r'=|\mathbf{x}'|$, $\cos\psi=\mathbf{x}\mathbf{x}'/rr'$. Then writing $\xi=1/r$, $\xi'=1/r'$ we have from (24″), (24‴), (31), and (31′)

$$V=\xi\int\rho\,d_3x'+\xi^2\int(\xi')^{-1}\cos\psi\,\rho\,d_3x'+\tfrac{1}{2}\xi^3\int(\xi')^{-2}\rho(3\cos^2\psi-1)\,d_3x'+O(k^2\xi^4) \tag{41}$$

$$W_\mu=\xi\int\rho u_\mu\,d_3x'+\xi^2\int(\xi')^{-1}\cos\psi\,\rho u_\mu\,d_3x'+\tfrac{1}{2}\xi^3\int[(\xi')^{-2}\rho u_\mu(3\cos^2\psi-1)]\,d_3x'+O(k^{5/2}\xi^4) \tag{41'}$$

$$\hat{T} = -\xi \int \tau \, d_3x' - \xi^2 \int (\xi')^{-1} \cos\psi \, \tau \, d_3x' - \tfrac{1}{2}\xi^3 \int [(\xi')^{-2}\tau(3\cos\psi - 1)] \, d_3x' + O(k^3\xi^4) \tag{41''}$$

where

$$\tau = S_{\sigma\sigma} - \rho u_\sigma u_\sigma \tag{42}$$

is the spatial part trace of the energy tensor.

Now, with the order of approximation considered, from (41) and (41″) we have

$$\begin{aligned} V + \hat{T} = \xi\Big(\int \rho \, d_3x' - \int \tau \, d_3x'\Big) &+ \xi^2\Big[\int (\xi')^{-1}\rho\cos\psi \, d_3x' - \int (\xi')^{-1}\tau\cos\psi \, d_3x'\Big] \\ &+ \tfrac{1}{2}\xi^3\Big[\int (\xi')^{-2}\rho(3\cos^2\psi - 1) \, d_3x' \\ &- \int (\xi')^{-2}\tau(3\cos^2\psi - 1) \, d_3x'\Big] + O(k^3\xi^4) \end{aligned} \tag{43}$$

or, equivalently,

$$V + \hat{T} = \tfrac{1}{2}\int (\rho - \tau) \, d_3x' + \mathbf{x}\xi^3 \int \mathbf{x}'(\rho - \tau) \, d_3x' + \tfrac{1}{2}\xi^3 \int (\xi')^{-2}(\rho - \tau)(3\cos^2\psi - 1) \, d_3x' + O(k^3\xi^4) \tag{44}$$

because

$$\xi^2\Big[\int (\xi')^{-1}(\mathbf{x}\mathbf{x}')(\xi\xi')\rho \, d_3x' - \int (\xi')^{-1}(\mathbf{x}\mathbf{x}')(\xi\xi')\tau \, d_3x'\Big] = \mathbf{x}\xi^3 \int \mathbf{x}'(\rho - \tau) \, d_3x' \tag{45}$$

Now the second term in (44) can be vanished introducing the mass center

$$\mathbf{R} = \frac{\int \mathbf{x}'(\rho - \tau) \, d_3x'}{\int (\rho - \tau) \, d_3x'} \tag{46}$$

and, as in Fock (1964), we then have

$$V+\hat{T}=\tfrac{1}{2}\int(\rho-\tau)\,d_3x'+\tfrac{1}{2}\xi^3\left[\int(\xi')^{-2}\rho(3\cos^2\psi-1)\,d_3x'\right.$$
$$\left.-\int(\xi')^{-2}\tau(3\cos^2\psi-1)\,d_3x'\right]+O(k^3\xi^4) \qquad (47)$$

Writing the last term of the right-hand side of the form

$$\tfrac{1}{2}\xi^3\left[\int(\xi')^{-2}[3(\mathbf{xx}')(\xi\xi')-1]\rho\,d_3x'-\int(\xi')^{-2}\left[3(\mathbf{xx}')(\xi\xi')-1\right]\tau\,d_3x'\right] \qquad (48)$$

or, equivalently, in the form

$$\tfrac{1}{2}\xi^3\left\{\int\tfrac{1}{2}\xi^2[3(x_1x_1'+x_2x_2'+x_3x_3')^2-(x_1^2+x_2^2+x_3^2)(x_1'^2+x_2'^2+x_3'^2)]\,dm\right.$$
$$\left.-\int\tfrac{1}{2}\xi^2[3(x_1x_1'+x_2x_2'+x_3x_3')^2-(x_1^2+x_2^2+x_3^2)(x_1'^2+x_2'^2+x_3'^2)]\,d\hat{t}\right. \qquad (49)$$

where $d\hat{t}=\tau\,d_3x'$, we see in the first term of (49) the moments of inertia

$$I_{x_1}=\int(x_2'^2+x_3'^2)\,dm,\qquad I_{x_2}=\int(x_1'^2+x_3'^2)\,dm$$
$$I_{x_3}=\int(x_1'^2+x_2'^2)\,dm \qquad (50)$$

and in the second we have

$$E_{x_1}=\int(x_2'^2+x_3'^2)\,d\hat{t},\qquad E_{x_2}=\int(x_1'^2+x_3'^2)\,d\hat{t}$$
$$E_{x_3}=\int(x_1'^2+x_2'^2)\,d\hat{t} \qquad (51)$$

which, by analogy with (51), we call moments of inertia for the stress.

On the other hand, if the massive body has an equatorial plane of symmetry, we can choose the axis Ox_1 and Ox_2 in it. This means that the quadratic forms Ix_ix_j and Ex_ix_j are reduced simultaneously to their diagonal form. Furthermore we have

$$I_{x_1}=I_{x_2}=I_e$$
$$E_{x_1}=E_{x_2}=E_e \qquad (52)$$

Then (49) can be written in the form

$$\tfrac{1}{2}\xi^5[x_1^2(I_{x_3}-I_e)+x_2^2(I_{x_3}-I_e)+x_3^2(I_{x_3}-I_e)$$
$$-x_1^2(E_{x_3}-E_e)-x_2^2(E_{x_3}-E_e)-x_3^2(E_{x_3}-E_e)] \qquad (53)$$

or, using spherical coordinates (r, θ, δ), in the form

$$\tfrac{1}{2}\xi^3(1-3\sin^2\delta)[(I_{x_3}-I_e)-(E_{x_3}-E_e)] \tag{54}$$

with that, on substituting in (47) this expression, we have

$$V+\hat{T}=m\xi-\hat{t}\xi+\tfrac{1}{2}\xi^3(1-3\sin^2\delta)[(I_{x_3}-I_e)-(E_{x_3}-E_e)]+O(k^3\xi^4) \tag{55}$$

or, what is the same,

$$V+\hat{T}=m\xi-\hat{t}\xi+\tfrac{1}{2}\xi^3(1-3x_3^2\xi^2)[(I_{x_3}-I_e)-(E_{x_3}-E_e)]+O(k^3\xi^4) \tag{55'}$$

In a similar way we can write (41′) in the form

$$W_\mu=\xi^2\int(\xi')^{-1}(\mathbf{xx}')(\xi\xi')\rho u_\mu\,d_3x' +\tfrac{1}{2}\xi^3\int(\xi')^{-2}\rho u_\mu[3(\mathbf{xx}')(\xi\xi')-1]\,d_3x' \tag{56}$$

Introducing finally (55′) and (56) in (30) we obtain definitely

$$\begin{aligned}\dot{u}_\rho=&[m\xi-\hat{t}\xi+\tfrac{1}{2}\xi^3(1-3x_3^2\xi^2)\{(I_{x_3}-I_e)-(E_{x_3}-E_e)\}]_{,\rho}\\&+[m\xi+\tfrac{1}{2}\xi^3(1-3x_3^2\xi^2)(I_{x_3}-I_e)]_{,\rho}\\&\times\{u^2-4[m\xi+\tfrac{1}{2}\xi^3(1-3x_3^2\xi^2)(I_{x_3}-I_e)]\}-4u_\rho\\&\times\{[m\xi+\tfrac{1}{2}\xi^3(1-3x_3^2\xi^2)(I_{x_3}-I_e)]_{,\mu}u_\mu\}\\&+4u_\mu\left\{\left[\xi^2\int(\mathbf{xx}')\xi'\rho u_\rho\,d_3x'\right]_{,\mu}-\left[\xi^2\int(\mathbf{xx}')\xi'\rho u_\mu\,d_3x'\right]_{,\rho}\right\}\\&+4u_\mu\left\{\int\tfrac{1}{2}\rho u_\rho[6(\mathbf{xx}')x'_\mu\xi^5-15(\mathbf{xx}')^2x'_\mu\xi^7-3(\xi')^{-2}x_\mu\xi^5]\right.\\&\left.-\int\tfrac{1}{2}\rho u_\mu[6(\mathbf{xx}')x'_\rho\xi^5-15(\mathbf{xx}')^2x'_\rho\xi^7-3(\xi')^{-2}x_\rho\xi^5]\right\}+O(k^3\xi^4)\end{aligned} \tag{57}$$

which are the equations of motion wanted.

5. THE PERTURBATION DUE TO STRESS

Writing equations (30) in the form

$$\dot{u}_\rho=V_{,\rho}+F_\rho \tag{30'}$$

we see in the first two terms the Newtonian equations of motion because V is the newtonian potential per unit mass. Following Synge (1969) we call to the rest

$$F_\rho=V_{,\rho}(u^2-4V)-4u_\rho\dot{V}+P_{,\rho}-Q_{,\rho}+4(W_{\rho,\mu}-W_{\mu,\rho})u_\mu+O(k^3) \tag{30''}$$

the relativistic perturbing force per unit mass. It must be noted that, as can be seen in (30″), this force not only depends on the classical potential V but also on the body's own stress and rotation. On the other hand, as it can be seen in (57), the perturbation term of F_ρ on a classical orbit due to them appears in these equations as gradient of the potential function

$$\hat{R} = -\hat{t}\xi + (\tfrac{1}{2}\xi^3 - \tfrac{3}{2}x_3^2\xi^5)[(I_{x_3} - I_e) - (E_{x_3} - E_e)] \tag{58}$$

Now in order to study its effect on an orbital motion we begin considering first an equatorial trajectory. Analytically the trajectory for a test particle can be obtained by means of equations (30), but as the field (25) is stationary these equations have the first integral of energy as all fields with this character. Furthermore, in our case also these equations have another first integral which is the one angular momentum because the generating body of the field has an axis of symmetry. Then, as these two first integrals are enough to determine an equatorial trajectory if, as we are supposing, the body has also an equatorial plane of symmetry, we can use these two integrals in order to determine the equation for an orbit of this type. The result is easy to obtain. We take the integral of energy for equations (30) which is (see Appendix A for its derivation)

$$\begin{aligned}(\tfrac{1}{2}u^2 - V) - \tfrac{1}{2}\kappa\tilde{J}S_{\sigma\sigma} + \tfrac{1}{2}\kappa\tilde{J}(\rho u^2) + 5V^2 + 6EV \\ = E - \tfrac{3}{2}E^2 + O(k^3)\end{aligned} \tag{59}$$

where V is given by (24″) and E is the constant which we can call total energy per unit mass of the test particle, and we take the integral of angular momentum which is (see Appendix B)

$$R^2\dot{\phi} = 4(x_1W_2 - x_2W_1) + h[1 - 4V + O(k^2)] \tag{60}$$

where W_μ is given by (24‴) and $h = A(1-E)$, where A is the angular momentum per unit mass. Then passing to polar coordinates r, ϕ, we write (59) and (60) in terms of them, and combining conveniently and in the usual way the resultant expressions, we have

$$\left(\frac{d\xi}{d\phi}\right)^2 = -F(\xi) \tag{61}$$

where $\xi = 1/r$ and

$$\begin{aligned}F(\xi) = \xi^2 - h^{-2}[2(E+V) + 2P - 2Q + 6V^2 + 4EV - 3E^2] \\ + h^{-3}[16(E+V)(x_1W_2 - x_2W_1)] + O(k^2)\end{aligned} \tag{61′}$$

with P and Q given as in (31) and (31′), respectively.

Now if one wishes to analyze the orbit we must study the advance of its apsidal line. As is known the formula for the general advance is given by

$$\varepsilon = 2\Delta\phi - 2\pi \tag{62}$$

where

$$\Delta\phi = \int_{\xi_1}^{\zeta_2} [-F(\xi)]^{-1/2}\, d\xi \tag{62'}$$

with $F(\xi)$ as in (61′) and ξ_1 and ξ_2 satisfying

$$F(\xi_1) = F(\xi_2) = 0, \qquad \xi_2 > \xi_1 > 0 \tag{62''}$$

Owing to the singularities in the integrand of (62′) it is necessary to introduce a new function $G(\xi)$ without singularities at the end of the interval (ξ_1, ξ_2) which is given by

$$G(\xi) = \frac{(B-B_1)(\xi-\xi_2)-(B-B_2)(\xi-\xi_1)}{(\xi-\xi_1)(\xi-\xi_2)(\xi_1-\xi_2)} \tag{63}$$

where

$$B(\xi) = h^{-2}[2(V-m\xi)+2P-2Q+6V^2+4EV-3E^2$$
$$-h^{-1}16(E+V)(x_1W_2-x_2W_1)]+O(k^2) \tag{63'}$$

and $B_1 = B(\xi_1)$, $B_2 = B(\xi_2)$. Then in terms of $G(\xi)$ we have

$$\Delta\phi = \int_{\xi_1}^{\xi_2} [(\xi_2-\xi)(\xi-\xi_1)]^{-1/2}[1-G(\xi)]^{-1/2}\, d\xi \tag{64}$$

Now in order for (62′) to be solved it is necessary to do some analytical simplifications in (41), (41′), and (41″). Thus if we take

$$\begin{aligned} V &= m\xi + \mu_3\xi^3 \\ W_\mu &= \xi^2 \int (\xi')^{-1}(\mathbf{xx}')(\xi\xi')\rho u_\mu\, d_3x' \\ P &= p_1\xi + p_3\xi^3 \\ Q &= q_1\xi + q_3\xi^3 \end{aligned} \tag{65}$$

where $\mu_3 = O(k^2)$ is the quadrupole potential, $p_1 = \text{const} = O(k^2)$, $p_3 = \text{const} = O(k^2)$, $q_1 = \text{const} = O(k^2)$ and $q_3 = \text{const} = O(k^2)$ [see (55) and (56)], then $B(\xi)$ can be written in the form

$$B(\xi) = b_0 + b_1\xi + b_2\xi^2 + b_3\xi^3 + \cdots + O(k^2) \tag{66}$$

where

$$b_0 = -3E^2h^{-2} \tag{66'}$$

$$b_1 = 2h^{-2}(p_1 - q_1 + 2mE - 4h^{-1}EL_3) \tag{66''}$$

$$b_2 = 6m^2h^{-2} - 8mL_3h^{-3} \tag{66'''}$$

$$b_3 = 2h^{-2}(\mu_3 + p_3 - q_3) \tag{66'v}$$

and L_3 is given as in (26). Then expanding $G(\xi)$ in (64) and taking into account (62) and (66) we have for ε the value

$$\varepsilon = \varepsilon_0 b_0 + \varepsilon_1 b_1 + \varepsilon_2 b_2 + \varepsilon_3 b_3 + \cdots + O(k^2) \tag{67}$$

where

$$\varepsilon_n = \int_{\xi_1}^{\xi_2} [(\xi_2 - \xi)(\xi - \xi_1)]^{-1/2} \frac{(\xi^n - \xi_1^n)(\xi - \xi_2) - (\xi^n - \xi_2^n)(\xi - \xi_1)}{(\xi - \xi_1)(\xi - \xi_2)(\xi_1 - \xi_2)} \, d\xi \tag{68}$$

Now, integrating here for $n = 0$, 1, 2, and 3 we have

$$\varepsilon_0 = 0, \qquad \varepsilon_1 = 0, \qquad \varepsilon_2 = \pi, \qquad \varepsilon_3 = 3\pi m h^{-2} + O(k) \tag{68$'$}$$

and so we have for the advance the expression

$$\begin{aligned} \varepsilon = {} & 6\pi m^2 h^{-2} - 8\pi m L_3 h^{-3} + 6\pi m \mu_3 h^{-4} \\ & + 6\pi m p_3 h^{-4} - 6\pi m q_3 h^{-4} + \cdots + O(k^2) \end{aligned} \tag{69}$$

As it can be seen in (69), here appear the known advances of Schwarzschild and rotation of the massive body together with the classical contribution due to its oblateness and also the one due to its stress. Furthermore it must be noted that, as it can be seen in (66′), although b_1 has a nonzero value, its corresponding term ε_1 in the expression (67) vanish, as we see from (68′). This means that, in contrast with the first term in the expansion of V, the first terms in the expansions of P and Q in (65) (and so, the first term in the expression of $\hat{T}$) do not give any contribution to the advance of perihelion for an equatorial orbit. Then in order to see what is their contribution for a general orbit let us use equations (57), and with these equations we shall use the theory of perturbations. With this purpose we introduce the parameters

$$K = \frac{3}{2} \frac{I_{x_3} - I_e}{m r_e^2}, \qquad K' = \frac{3}{2} \frac{E_{x_3} - E_e}{m r_e^2} \tag{70}$$

of which the first is proportional to the known dimensionless constant J_2 (the parameter associated to the quadrupole of the body). As before m is the mass of the body and r_e its equatorial radius. Then, using the third Kepler law for the osculating orbit of period T associated to the classical equations contained in (57), we write (58) in the form

$$\hat{R} = -\hat{t}\xi + \xi^3(\tfrac{1}{3} - x_3^2 \xi^2)(K - K') n_0^2 a^3 r_e^2 \tag{71}$$

(where $n_0 = 2\pi/T$ and a is the semimajor axis of the classical orbit), or in terms of its elements

$$\hat{R} = -\hat{t}\xi + \xi^3(\tfrac{1}{3} - \sin^2 i \sin^2 u)(K - K') n_0^2 a^3 r_e^2 \tag{71$'$}$$

where i is the inclination, f the true anomaly, ω the argument of perihelion, and $u = \omega + f$.

Now, in order to determine the secular variation, V_s, for the orbit, we average (71′) by means of the integral

$$V_s = \frac{1}{T}\int_0^T \hat{R}\,dt \tag{72}$$

The result is

$$V_s = -\hat{t}a^{-1} + (K - K')n_0^2 r_e^2(\tfrac{1}{3} - \tfrac{1}{2}\sin^2 i)(1 - e^2)^{-3/2} \tag{73}$$

where e is the eccentricity for the osculating orbit. As can be seen in (73), the potential function which gives the secular perturbation due to the stress depends only on the inclination, eccentricy, and semimajor axis of the osculating orbit.

Now in order to obtain the evolution of the orbital elements we apply the Lagrange equations for the perturbed motion (see, e.g., equations (59), Chap. IX of Stiefel and Scheifele (1971). From these equations we deduce that (73) does not induce variations in the variables i, e, and a previously mentioned. The only parameters which change due to V_s are the true anomaly, M, the longitude of the ascending node, Ω, and the argument of perihelion ω, following the equations

$$\dot{M} = n_0 - \frac{K - K'}{a^2} r_e^2 n_0(1 - \tfrac{3}{2}\sin^2 i)(1 - e^2)^{-3/2} - 2\frac{\hat{t}}{n_0 a^3} \tag{74}$$

$$\dot{\Omega} = -\frac{K - K'}{a^2} r_e^2 n_0 \cos i(1 - e^2)^{-2} \tag{74′}$$

$$\dot{\omega} = \frac{K - K'}{a^2} r_e^2 n_0(2 - \tfrac{5}{2}\sin^2 i)(1 - e^2)^{-2} \tag{74″}$$

Then integrating (74), (74′), and (74″) we obtain

$$\delta M = \left[n_0 - \frac{K - K'}{a^2} r_e^2 n_0\left(1 - \frac{3}{2}\sin^2 i\right)(1 - e^2)^{-3/2} - \frac{2\hat{t}}{n_0 a^3}\right]t \tag{75}$$

$$\delta\Omega = -\frac{K - K'}{a^2} r_e^2 n_0 \cos i\,(1 - e^2)^{-2} t \tag{75′}$$

$$\delta\omega = \frac{K - K'}{a^2} r_e^2 n_0(2 - \tfrac{5}{2}\sin^2 i)(1 - e^2)^{-2} t \tag{75″}$$

Now since (75), (75′), and (75″) give the changes of M, Ω and ω during t seconds, it follows that, after a period T, these changes are finally

$$\frac{\delta M}{T} = n_0 - \frac{K - K'}{a^2} r_e^2 n_0(1 - \tfrac{3}{2}\sin^2 i)(1 - e^2)^{-3/2} - \frac{2\hat{t}}{n_0 a^3} \tag{76}$$

$$\frac{\delta\Omega}{T}=-\frac{K-K'}{a^2}r_e^2 n_0 \cos i\,(1-e^2)^{-2} \tag{76'}$$

$$\frac{\delta\omega}{T}=\frac{K-K'}{a^2}r_e^2 n_0(2-\tfrac{5}{2}\sin^2 i)(1-e^2)^{-2} \tag{76''}$$

As can be seen in (76), (76′), and (76″), only the classical prediction for the mean anomaly is affected by the first terms in the expansions of P and Q in (65). Indeed if these two terms are not taken into account then the value of i for which

$$\frac{\delta M}{T}=n_0 \tag{77}$$

is

$$i=\arcsin\sqrt{\tfrac{2}{3}} \tag{78}$$

whereas when they are taking into account we have for this value of i

$$\frac{\delta M}{T}=n_0-2\frac{\hat{t}}{n_0 a^3} \tag{79}$$

On the other hand, for any other value of i, it must be noted that the nonspherical distribution of the stress also manifests itself changing the classical parameter K by $K-K'$ in all equations (for comparison with the classical predictions see, e.g., equations (69), Chap. IX of Stiefel and Scheifele, 1971). In particular, when $i=0$ we have an advance of perihelion which corresponds to the fourth and fifth terms obtained in (69). In any case all these contributions are, according to (57), $O(k^2\xi^3)$.

APPENDIX A: THE INTEGRAL OF ENERGY [see equation (59)]

Let g_{ab} be the metric,

$$g_{ab}=\delta_{ab}+\gamma_{ab} \tag{A1}$$

where

$$\begin{aligned}\gamma_{\mu\nu}&=2(V+V^2)\delta_{\mu\nu}-2\kappa\tilde{J}[(\rho u_\mu u_\nu)^*]\\&\quad+4\tilde{J}[\tfrac{1}{2}\kappa S^*_{\mu\nu}+(V^2{}_{,\mu\nu}-V_{,\mu}V_{,\nu}]+O(k^3)\\ \gamma_{\mu 4}&=-2\kappa i\tilde{J}(\rho u_\mu)-8i\tilde{J}(V_{,\sigma}W_{\sigma,\mu}-V_{,\mu\sigma}W_\sigma\\&\quad+W_\mu\Delta V-V\Delta W_\mu)+O(k^{7/2})\\ \gamma_{44}&=-2(V-V^2)-\kappa\tilde{J}S_{\sigma\sigma}+\kappa\tilde{J}(\rho u_\sigma u_\sigma)+O(k^3)\end{aligned} \tag{A2}$$

with

$$V=-4\pi\tilde{J}(\rho),\qquad W_\mu=-4\pi\tilde{J}(\rho u_\mu),\qquad \kappa=8\pi$$

$$S^*_{\mu\nu}=S_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}S_{\sigma\sigma} \tag{A3}$$

$$(\rho u_\mu u_\nu)^*=(\rho u_\mu u_\nu)-\tfrac{1}{2}\delta_{\mu\nu}(\rho u_\sigma u_\sigma)$$

and

$$\tilde{J}f(\mathbf{x},t)=-(4\pi)^{-1}\int f(\mathbf{x}',t)|\mathbf{x}-\mathbf{x}'|^{-1}\,d_3x' \tag{A4}$$

Further

$$\frac{\partial u}{\partial t}=0,\qquad \frac{\partial \rho}{\partial t}=0,\qquad \frac{\partial S_{\mu\nu}}{\partial t}=0 \tag{A5}$$

and

$$\rho=O(k),\qquad u_\mu=O(k^{1/2}),\qquad S_{\mu\nu}=O(k^2),\qquad \frac{\partial}{\partial t}=O(k^{1/2}) \tag{A6}$$

Because of (A5), the Lagrange equations, which are equivalents to equations (32),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_\mu}-\frac{\partial L}{\partial x_\mu}=0 \tag{A7}$$

have the integral

$$L-\dot{x}_\mu\frac{\partial L}{\partial \dot{x}_\mu}=1+E \tag{A8}$$

From (A8) we deduce

$$L^{-1}\left(L^2-\frac{1}{2}\dot{x}_\mu\frac{\partial L^2}{\partial \dot{x}_\mu}\right)-1=E \tag{A9}$$

As

$$L=(-g_{\mu\nu}\dot{x}_\mu\dot{x}_\nu-2ig_{\mu4}\dot{x}_\mu+g_{44})^{1/2} \tag{A10}$$

we deduce from (A9)

$$L^{-1}(-ig_{\mu4}\dot{x}_\mu+g_{44})-1=E \tag{A11}$$

Now, taking into account (A6), (A2), and (A3), we have

$$\gamma_{\mu\nu}=O(k),\qquad \gamma_{\mu u}=O(k^{3/2}),\qquad \gamma_{44}=O(k) \tag{A12}$$

and then, carrying (A1) to (A11), it results in

$$E=\tfrac{1}{2}(u^2+\gamma_{44})+\tfrac{1}{2}\gamma_{\mu\nu}u_\mu u_\nu+\tfrac{1}{8}(u^2-\gamma_{44})(3u^2+\gamma_{44})+O(k^3) \tag{A13}$$

As the principal part of (A14) is

$$\tfrac{1}{2}u^2 - V \tag{A14}$$

we have from (A13)

$$(\tfrac{1}{2}u^2 - V) - \tfrac{1}{2}\kappa\tilde{J}S_{\sigma\sigma} - \tfrac{1}{2}\kappa\tilde{J}(\rho u^2) + \tfrac{3}{8}u^4 + \tfrac{3}{2}u^2 V + \tfrac{1}{2}V^2 + O(k^3) = E \tag{A15}$$

because

$$\tfrac{1}{2}u^2 - V + O(k^2) = E \tag{A16}$$

Finally from (A15) we have

$$(\tfrac{1}{2}u^2 - V) - \tfrac{1}{2}\kappa\tilde{J}S_{\sigma\sigma} + \tfrac{1}{2}\kappa\tilde{J}(\rho u^2) - SV^2 + 6EV = E - \tfrac{3}{2}E^2 + O(k^3) \tag{A17}$$

which is the integral of energy.

APPENDIX B: THE INTEGRAL OF ANGULAR MOMENTUM [see equation (60)]

As the azimuthal angle ϕ is a cyclic coordinate, we have

$$\frac{\partial L}{\partial \dot{\phi}} = -A \tag{B1}$$

Using Cartesian coordinates, (B1) is reduced to

$$x_1 \frac{\partial L}{\partial \dot{x}_2} - x_2 \frac{\partial L}{\partial \dot{x}_1} = -A \tag{B2}$$

From (B2) we deduce

$$-\frac{1}{2}L^{-1}\left[x_1 \frac{\partial L^2}{\partial \dot{x}_2} - x_2 \frac{\partial L^2}{\partial \dot{x}_1}\right] = A \tag{B3}$$

and from (B3) we have

$$L^{-1}[x_1 g_{2\mu}\dot{x}_\mu - x_2 g_{1\mu}\dot{x}_\mu + i(x_1 g_{24} - x_2 g_{14})] = A \tag{B4}$$

Taking into account (A2), (A3), and (A6), from (B4) we deduce

$$(1 + \tfrac{1}{2}u^2 - \tfrac{1}{2}\gamma_{44})(x_1 u_2 - x_2 u_1) - (x_1 \gamma_{2\mu} u_\mu - x_2 \gamma_{1\mu} u_\mu) + i(x_1 \gamma_{24} - x_2 \gamma_{14}) + O(k^{5/2}) = A \tag{B5}$$

and substituting (A2) in (B5) we have

$$A = (1 + \tfrac{1}{2}u^2 + 3V)(x_1 u_2 - x_2 u_1) - 4(x_1 W_2 - x_2 W_1) + O(k^{5/2}) \tag{B6}$$

(B6) is reduced, taking into account (A16), to

$$(1+E+4V)(x_1u_2-x_2u_1)-4(x_1W_2-x_2W_1)+O(k^{5/2})=A \qquad \text{(B7)}$$

which in cylindrical coordinates is reduced to

$$(1+E+4V)R^2\dot{\phi}-4(x_1W_2-x_2W_1)+O(k^{5/2})=A \qquad \text{(B8)}$$

But as

$$AE=ER^2\dot{\phi}+O(k^{5/2}) \qquad \text{(B9)}$$

we have

$$(1+4V)R^2\dot{\phi}-4(x_1W_2-x_2W_1)+O(k^{5/2})=A(1-E)=h \qquad \text{(B10)}$$

and finally we have

$$R^2\dot{\phi}=4(x_1W_2-x_2W_1)+h[1-4V+O(k^2)] \qquad \text{(B11)}$$

which is the integral of angular momentum.

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